



Stability and asymptotic behavior of perturbed difference systems¹

Rigoberto Medina

Departamento de Ciencias Exactas. Universidad de Los Lagos, Casilla 933, Osorno, Chile

Received 20 March 1998

Abstract

The asymptotic behavior of the solutions of a class of functional-difference equations is studied. The results obtained, which are applied to delay difference equations and summary difference equations, give good estimates and explicit radius of attraction. © 1998 Elsevier Science B.V. All rights reserved.

AMS classification: 39A10; 39A11

Keywords: Delay difference equations; Summary difference equations; Radius of attraction

1. Introduction

Consider the nonlinear difference system

$$x(n+1) = f(n, x(n)), \quad (1)$$

along with its associated variational system

$$z(n+1) = f_x(n, x(n, n_0, x_0))z(n) \quad (2)$$

and the perturbed systems

$$y(n+1) = f(n, y(n)) + g(n, y(n), Ty(n)), \quad (3)$$

where

$$f : N(n_0) \times D \rightarrow \mathbb{R}^m \quad \text{and} \quad g : N(n_0) \times D \times D \rightarrow \mathbb{R}^m$$

are two functions, and $D \subseteq \mathbb{R}^m$ is a domain with $0 \in D$, $N(n_0) = \{n_0, n_0 + 1, \dots, n_0 + k, \dots\}$ (n_0 a nonnegative integer), and let $f_x = \partial f / \partial x$ exist and be continuous and invertible on $N(n_0) \times \mathbb{R}^m$,

¹ Work partially supported by FONDAP de Matemáticas Aplicadas and FONDECYT No. 1970427.

$f(n, 0) = 0$ and $g(n, 0, 0) = 0$ for all $n \geq n_0$ and $x(n) = x(n, n_0, x_0)$ is the solution of Eq. (1) with $x(n_0, n_0, x_0) = x_0$. T is an operator mapping $S(N(n_0), \mathbb{R}^m)$ into $S(N(n_0), \mathbb{R}^m)$. Here, $S(N(n_0), \mathbb{R}^m)$ represents the set of all sequences $y: N(n_0) \rightarrow \mathbb{R}^m$. If we impose on T various meanings, various types of equations will appear. For example, some of them have the following form:

$$y(n+1) = f(n, y(n)) + g(n, y(n), y(n-\tau)),$$

$$y(n+1) = f(n, y(n)) + g(n, y(n), y(h(n))),$$

$$y(n+1) = f(n, y(n)) + g\left(n, y(n), \sum_{i=n_0}^{n-1} k(n, i, y(i))\right),$$

and so on.

We shall discuss the question under what conditions Eq. (3) preserves some properties of Eq. (1) related to the asymptotic behavior of the solutions. We shall investigate the properties: boundedness, exponential asymptotic stability in variation, asymptotical stability and convergence to zero.

In the present paper, we continue the research initiated in [6–8] addressing our study to functional difference equations.

Our results, though different in their approach, compare favorably with results in [3, 5, 10], and in general they have very little overlapping with those. In fact, on the one hand, we make precise the initial conditions (the radius of attraction) for which the solutions tend to zero as $n \rightarrow \infty$ or are bounded. On the other hand, we obtain good estimates for the solutions of Eq. (3) depending on the sum-norm (l_1 -norm) of the variable coefficients of the perturbation g .

2. Preliminaries

In order to establish our main results, we shall use the following:

Lemma 1 (Lakshmikantham and Trigiante [4, Lemma 4.6.2]). *Assume that $f: N(n_0) \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ and f possess partial derivatives on $N(n_0) \times \mathbb{R}^m$. Let the solution $x(n) = x(n, n_0, x_0)$ of system (1) exist for $n \geq n_0$ and let*

$$H(n, n_0, x_0) = \frac{\partial f(n, x(n, n_0, x_0))}{\partial x}.$$

Then,

$$\Phi(n, n_0, x_0) = \frac{\partial x(n, n_0, x_0)}{\partial x_0}$$

exists and is the solution of

$$\Phi(n+1, n_0, x_0) = H(n, n_0, x_0)\Phi(n, n_0, x_0),$$

$$\Phi(n_0, n_0, x_0) = I.$$

If $x(n)$ and $y(n)$ are the solutions of Eqs. (1) and (3), respectively, and $x(n_0) = y(n_0)$, then the following analogue of Alekseev's formula holds [2],

$$y(n) = x(n) + \sum_{l=n_0}^{n-1} \int_0^1 \Phi(n, l+1, u(y(l), \tau)) \times g(l, y(l), Ty(l)) d\tau, \quad (4)$$

where $\Phi(n, n_0, x(n, n_0, x_0))$ is the fundamental matrix of system (2), and

$$u(y(n), \tau) = f(n, y(n)) + \tau g(n, y(n), Ty(n)), \quad \tau \in [0, 1].$$

A generalization of (4) can be found in [10].

Definition 1. The solution $x=0$ of Eq. (1) is said to be exponentially asymptotically stable in variation if there exist two positive constants M and η with $0 < \eta < 1$, such that

$$\|\Phi(n, n_0, x_0)\| \leq M\eta^{n-n_0} \quad \text{for } n_0 \leq n < \infty \quad (5)$$

and $x_0 \in D$.

Here $\|\cdot\|$ denotes a usual norm.

Now, we need a “solution” of the functional inequalities

$$u(n) \leq c + \sum_{i=1}^p \left[\sum_{k=0}^{n-1} \lambda_i(k) \omega_i(u(k)) \right], \quad p \in \mathbb{N} \quad (6)$$

and

$$u(n) \leq c + \sum_{k=n_0}^{n-1} \lambda_1(k) \omega_1(u(k)) + \sum_{k=n_0}^{n-1} \lambda_2(k) \omega_2 \left[\sum_{j=n_0}^{k-1} \lambda_3(j) \omega_3(u(j)) \right], \quad (7)$$

where

(H₁) the functions $\omega_i: [d, \infty) \rightarrow [0, \infty)$, ($i=1, 2, \dots, p$) are continuous and nondecreasing, $\omega_i(u) > 0$ for $u > d$ and ω_{i+1}/ω_i ($i=1, 2, \dots, p-1$) are nondecreasing on (d, ∞) .

(H₂) $u: \mathbb{N} \rightarrow [d, \infty)$ and $\lambda_i: \mathbb{N} \rightarrow [0, \infty)$, are functions, c is a constant such that $c > d$.

We define the functions:

- (i) $W_i(u) = \int_{u_i}^u ds/\omega_i(s)$, $u > 0$, $u_i > 0$ ($i=1, 2, \dots, p$) and W_i^{-1} is their inverse function.
- (ii) $\phi_0(u) = u$ and

$$\phi_i(u) = \phi_i \circ \phi_{i-1} \circ \dots \circ \phi_1, \quad i=1, 2, \dots, p, \quad (8)$$

where $\phi_i(u) = W_i^{-1}[W_i(u) + \alpha_i]$, $\alpha_i \geq 0$ is a constant.

Thus, we can establish the following theorem:

Theorem A. (Medina and Pinto [8]). Let $d \in \mathbb{R}$ and assume (H₁) and (H₂) hold. Let $m \in \mathbb{N}$ such that

$$\alpha_i(m) =: \sum_{k=1}^m \lambda_i(k) \leq \int_{\phi_{i-1}(c)}^{\infty} \frac{ds}{\omega_i(s)}, \quad i=1, 2, \dots, p, \quad (9)$$

where the functions ϕ_i ($i=0, 1, \dots, p-1$) are given in (8) with $\alpha_i = \alpha_i(m)$.

If the sequence u satisfies the inequality (6), then

$$u(n) \leq W_p^{-1} \left[W_p(\varphi_{p-1}(c)) + \sum_{k=1}^{n-1} \lambda_p(k) \right], \quad (10)$$

for any $n \leq m$.

Theorem B. (Medina and Pinto [8]). *Under the conditions of Theorem A, if (7) holds then (10) is valid for $p=3$, where m satisfies (9) for $p=3$.*

We remark that m in (9) can be taken as large as possible if

$$\int_1^\infty \frac{ds}{\omega_i(s)} = \infty \quad (i=1, 2, \dots, p) \quad (11)$$

which implies that any φ_k (and ϕ_k) is defined for all u and $m \geq n_0$. Thus, (10) is valid for all $n \geq n_0$ and $c \geq 0$.

The dual condition to (11), namely,

$$\int_{0^+}^1 \frac{ds}{\omega_i(s)} = \infty \quad (i=1, 2, \dots, p), \quad (12)$$

implies that any φ_k (and ϕ_k) is defined for u small enough and any $m \geq n_0$. Thus, (10) is valid for any $n \geq n_0$ if c is small enough. Moreover, (12) implies the stability property

$$\varphi_k(0^+) = 0 \quad (k=1, 2, \dots, p). \quad (13)$$

Furthermore, the inequalities (9) allow us to compute m (see [8]). In the following, we consider the functions φ_i ($i=1, 2, \dots, p$) given by (8) with $m = \infty$.

Lemma 2 (Bainov and Simeonov [1, Lemma 11.1]). *Let $q \in [0, 1)$, $\gamma_k \geq 0$ and $\lim_{k \rightarrow \infty} \gamma_k = 0$ or $\sum_{k=1}^\infty \gamma_k < \infty$. Then,*

$$\lim_{k \rightarrow \infty} \left(\sum_{i=1}^k q^{k-i} \cdot \gamma_i \right) = 0.$$

3. Main results

We are now in a position to establish our main results.

Theorem 1. *Let ω_i ($i=1, 2, \dots, p$) be as in Theorem A. Suppose that*

- (i) λ_i ($i=1, 2, \dots, p$) are nonnegative sequences on $N(n_0)$ and $\lambda_i \in l_1(N(n_0))$, (that is, absolutely summable).
- (ii) for $n \geq l+1 \geq n_0$, we have

$$\int_0^1 \|\Phi(n, l+1, u(y(l), \tau))g(j, y(j), Ty(j))\| d\tau \leq \sum_{i=1}^p \lambda_i(l) \omega_i(\|y(l)\|),$$

and

(iii) there exists a positive constant c such that

$$\sum_{l=n_0}^{\infty} \lambda_p(l) < \int_{\varphi_{p-1}(c)}^{\infty} \frac{ds}{\omega_p(s)}.$$

Then, for each bounded solution $x(n, n_0, x_0)$ of Eq. (1) such that $\|x(n, n_0, x_0)\| \leq c$ for $n \geq n_0$, the solution $y(n, n_0, x_0)$ of Eq. (3) is defined and bounded on $N(n_0)$. Moreover,

$$\|y(n, n_0, x_0)\| \leq \varphi_p(\|x\|_{\infty}). \quad (14)$$

Proof. Let $y(n) = y(n, n_0, x_0)$ be a solution of the perturbed Eqs. (3) for $n \geq n_0$. By the discrete Alekseev's formula (4), the solution $y(n)$ of Eq. (3) satisfies Eq. (4).

Hence,

$$\|y(n)\| \leq \|x\|_{\infty} + \sum_{i=1}^p \sum_{j=n_0}^{n-1} \lambda_i(j) \omega_i(\|y(j)\|),$$

where $\|x\|_{\infty} = \sup\{|x(n)|: n \in N(n_0)\}$. Thus, for all $n \geq n_0$ from Theorem A we have

$$\begin{aligned} \|y(n)\| &\leq W_p^{-1} \left[W_p(\varphi_{p-1}(c)) + \sum_{j=n_0}^{n-1} \lambda_p(j) \right] \\ &\leq \varphi_p(\|x\|_{\infty}). \end{aligned} \quad (15)$$

Condition (iii) implies that $\varphi_p(c) < \infty$. Consequently, $\varphi_p(\|x\|_{\infty}) \leq \varphi_p(c) < \infty$. Then, for any n fixed,

$$\int_0^1 \Phi(n, j+1, u(y(j), \tau)) g(j, y(j), Ty(j)) d\tau$$

is absolutely summable as a function of j . Thus, it follows that the solution $y(n)$ of Eq. (3) is defined and bounded on $N(n_0)$. \square

Remark 1. The method used in Theorem 1 can be applied to delay-differential equations (see [9]) and, in general, to those difference equations satisfying

$$\begin{aligned} &\int_0^1 \|\Phi(n, j+1, u(y(j), \tau)) g(j, y(j), Ty(j))\| d\tau \\ &\leq \sum_{i=1}^p \lambda_i(j) \omega_i(\|y\|_j) \quad \text{for } n \geq j+1 \geq n_0, \end{aligned}$$

where for $n^* = n - r$, $r \in \mathbb{N}$ constant, $\|y\|_n = \sup_{j \in I_n} |y(j)|$, $I_n = \{n^*, n^* + 1, \dots, n\}$. In fact, in this last case, from (4) we find

$$\|y\|_n \leq \|x\|_{\infty} + \sum_{i=1}^p \sum_{j=n_0}^{n-1} \lambda_i(j) \omega_i(\|y\|_j).$$

Hence, applying Theorem A to $u(n) = \|y\|_n$, we can establish Theorem 1 for this kind of equations.

Remark 2. (a) If (11) holds, then condition (iii) of Theorem 1 is satisfied for all $c > 0$.

(b) If (12) holds, then there always exists c small enough satisfying condition (iii).

(c) Finally, in the case $1/\omega_i \in L_1((0, \infty))$ (Lebesgue integrables), ($i = 1, 2, \dots, p$) the inequality

$$\sum_{j=n_0}^{\infty} \lambda_i(j) \geq \int_0^{\infty} \frac{ds}{\omega_i(s)}$$

for some i implies that there is not $c > 0$ satisfying condition (iii) of Theorem 1. Otherwise, there always exists c small enough satisfying condition (iii). In every case, the biggest c satisfying condition (iii) is

$$c = \varphi_p^{-1}(\infty). \quad (16)$$

(see [8]). Thus, we can establish.

Corollary 1. (A) If (11) holds, then the statement of Theorem 1 is valid for every solution.

(B) If (12) holds, then the statement of Theorem 1 is valid only for x such that $\|x\|_{\infty}$ is small enough, precisely for $\|x\|_{\infty} < \varphi_p^{-1}(\infty)$.

Remark 3. Equation $y(n+1) = y(n) + (e-1)/e^n y^2(n)$, $y(1) = n_0$, $n \geq 1$ and its solution $y(n) = e^n$, shows that the statement of Theorem 1 is not valid for arbitrary solutions. In fact, the equation

$$x(n+1) = x(n)$$

has the solution $x(n, n_0, x_0) = x_0$, for $n \geq n_0 \geq 1$. Moreover, $\omega_1(u) = u^2$, $\lambda_1(n) = (e-1)/e^n \in l_1 N(1)$.

Thus, the condition

$$\sum_{k=1}^{\infty} \lambda_1(k) < \int_0^{\infty} \frac{du}{u^2}$$

is only valid for $c < 1/(e-1)$.

Theorem 2. Assume that

(I) ω_i ($i = 1, 2, 3$) and λ_i ($i = 1, 2, 3$) satisfy Theorem 1.

(II) for $n_0 \leq j \leq n < \infty$ and $y: N(n_0) \rightarrow \mathbb{R}^m$, we have

$$\int_0^1 \|\Phi(n, j+1, u(y(j), \tau))g(j, y(j), Ty(j))\| d\tau$$

$$\leq \lambda_1(j)\omega_1(\|y(j)\|) + \lambda_2(j)\omega_2\left(\sum_{l=n_0}^{j-1} \lambda_3(l)\omega_3(\|y(l)\|)\right),$$

and

(III) there is a positive constant c such that

$$\sum_{j=n_0}^{\infty} \lambda_3(j) < \int_{\varphi_2(c)}^{\infty} \frac{ds}{\omega_3(s)}.$$

Then, for each bounded solution $x(n, n_0, x_0)$ of Eq. (1) such that $\|x(n, n_0, x_0)\| \leq c$ for $n \geq n_0$, the corresponding solution $y(n, n_0, x_0)$ of Eq. (3) is defined and bounded on $N(n_0)$ and

$$\|y(n, n_0, x_0)\| \leq \varphi_3(\|x\|_\infty). \quad (17)$$

Proof. Let $y(n) = y(n, n_0, x_0)$ be a solution of the perturbed Eq. (3), for $n \geq n_0$. By the discrete Alekseev's formula (4), the solution $y(n)$ of Eq. (3) satisfies Eq. (4). Thus, using (I) and (II), it follows that

$$\|y(n)\| \leq \|x\|_\infty + \sum_{j=n_0}^{n-1} \left[\lambda_1(j) \omega_1(\|y(j)\|) + \lambda_2(j) \omega_2 \left(\sum_{l=n_0}^{j-1} \lambda_3(l) \omega_3(\|y(l)\|) \right) \right].$$

Using Theorem B and proceeding analogously to Theorem 1, we can prove that $y(n, n_0, x_0)$ is defined and bounded on $N(n_0)$. \square

Now, we introduce the following condition:

(H₃) $u(y(n), \tau) = f(n, y(n)) + \tau \cdot g(n, y(n), Ty(n)) \in D$ for any $n \in N(n_0)$, $y \in D$ and $\tau \in [0, 1]$.

Theorem 3. Assume that the condition (H₃) holds, and that the null solution of Eq. (1) is exponentially asymptotically stable in variation. Moreover, suppose that the hypotheses of Theorem 2 hold, where (II) is replaced by

(II)' for $j \geq n_0$ and $y: N(n_0) \rightarrow \mathbb{R}^m$, we have

$$\|g(j, y(j), Ty(j))\| \leq \lambda_1(j) \omega_1(\|y(j)\|) + \lambda_2(j) \omega_2 \left(\sum_{l=n_0}^{j-1} \lambda_3(l) \omega_3(\|y(l)\|) \right).$$

Then, every solution y of Eq. (3) which satisfies $\|y(n_0)\| \leq cM^{-1}$, tends to zero as $n \rightarrow \infty$, and satisfies

$$\|y(n, n_0, y_0)\| \leq \varphi_3(M\|y_0\|). \quad (18)$$

Furthermore, the zero solution of Eq. (3) is asymptotically stable if (11) holds.

Proof. From Alekseev's formula (4), and (5) it follows that

$$\|y(n)\| \leq M\|x_0\| + \sum_{j=n_0}^{n-1} \int_0^1 \|\Phi(n, j+1, u(y(j), \tau))g(j, y(j), Ty(j))\| d\tau.$$

By (5), the matrix $\Phi(\cdot)$ is bounded on $N(n_0)$. Therefore, by (II)' the hypotheses of Theorem 2 are satisfied and then $y(n)$ is bounded on $N(n_0)$ and it verifies (18). Hence, we obtain that $\tilde{g}(j) =: g(j, y(j), Ty(j)) \in l_1(N(n_0))$.

Finally, to end the proof of the theorem, we need to show that

$$\lim_{n \rightarrow \infty} \sum_{j=n_0}^{n-1} \int_0^1 \|\Phi(n, j+1, u(y(j), \tau))\| \|\tilde{g}(j)\| d\tau = 0. \quad (19)$$

To prove (19), we make use of Lemma 2 ([2]). In fact, from Lemma 2, we have that

$$\begin{aligned} & \sum_{j=n_0}^{n-1} \int_0^1 \|\Phi(n, j+1, u(y(j), \tau))\| \|\tilde{g}(j)\| \, d\tau \\ & \leq M \cdot \sum_{j=n_0}^{n-1} \eta^{n-(j+1)} \|\tilde{g}(j)\| \\ & = M \cdot \sum_{j=n_0}^{n-1} \eta^{(n-1)-j} \|\tilde{g}(j)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Finally, the asymptotical stability of the zero solution of Eq. 3 follows from (13) and (18). Thus, the result follows. \square

4. Examples

Now, we will illustrate Theorems 1–3 showing explicitly the radius of attraction, that is, we make precise the initial conditions and the estimates for which the solutions work.

Example 1. Consider the summary difference equation

$$y(n+1) = h(n)k(y(n)) + \lambda_1(n)y(n) + \lambda_2(n) \sum_{i=n_0}^n \lambda_3(i)y^{\gamma}(i), \quad (20)$$

where $\gamma \geq 1$; λ_i ($i = 1, 2, 3$) are absolutely summable functions on $N(n_0)$, and $h, k: N(n_0) \rightarrow \mathbb{R}$ are functions such that the null solution of

$$x(n+1) = h(n)k(x(n)), \quad x(n_0) = x_0$$

is exponentially asymptotically stable in variation.

On the other hand, we have

- (A) $\omega_1(u) = \omega_2(u) = u$, $\omega_3(u) = u^{\gamma}$. Then, condition (i) of Theorem 1 and condition (II)' of Theorem 3 are automatically verified.
- (B) To verify condition (III) of Theorem 2,

$$\sum_{j=n_0}^{\infty} \lambda_3(j) < \int_{\varphi_2(c)}^{\infty} \frac{ds}{\omega_3(s)},$$

we distinguish two cases:

- (a) $\gamma = 1$. In this case, condition (III) is satisfied for any $c > 0$ since $\lambda_3 \in l_1(N(n_0))$ and the integral $\int_{\varphi_2(c)}^{\infty} ds/\omega_3(s) = +\infty$.
- (b) $\gamma > 1$. In this case, it is necessary to choose the correct constant c . We have that

$$\int_{\varphi_2(c)}^{\infty} \frac{ds}{s^{\gamma}} = \frac{-\varphi_2(c)^{1-\gamma}}{1-\gamma},$$

and if $\alpha_i = \sum_{j=n_0}^{\infty} \lambda_i(j)$; ($i = 1, 2, 3$), then condition (III) is equivalent to the inequality

$$\alpha_3 < \frac{\varphi_2(c)^{1-\gamma}}{\gamma-1}. \quad (21)$$

Since φ_2 is a monotone function and by (13), $\lim_{u \rightarrow 0^+} \varphi_2(u) = 0$.

Hence, choosing c small enough, we will get that (21) be satisfied. Solving the equation

$$\alpha_3 = \frac{\varphi_2(c^*)^{1-\gamma}}{\gamma-1},$$

we obtain

$$c^* = \varphi_2^{-1}(1^{-\gamma} \sqrt{\alpha_3(\gamma-1)}).$$

For determining c^* , we will calculate φ_2^{-1} explicitly. By definition, we have that

$$\varphi_1(u) = W_1^{-1}[W_1(u) + \alpha_1] = ue^{\alpha_1}$$

and

$$\varphi_2(u) = W_2^{-1}[W_2(\varphi_1(u)) + \alpha_2] = \varphi_1(u)e^{\alpha_2} = ue^{(\alpha_1+\alpha_2)}.$$

Thus,

$$\varphi_2^{-1}(u) = ue^{-(\alpha_1+\alpha_2)}.$$

We therefore have

$$c^* = e^{-(\alpha_1+\alpha_2)} 1^{-\gamma} \sqrt{\alpha_3(\gamma-1)}.$$

Then, taking $c \leq c^*$, condition (21) is satisfied.

We get from Theorem (3) that every solution $y(n, n_0, x_0)$ tends to zero as $n \rightarrow \infty$ if $\|x_0\| < c^* \cdot M$, that is, if

$$\|x_0\| < M e^{-(\alpha_1+\alpha_2)} 1^{-\gamma} \sqrt{\alpha_3(\gamma-1)}, \quad (22)$$

which gives an explicit radius of attraction. Moreover, the zero solution of Eq. (3) is asymptotically stable, and the following estimate is valid:

$$\|y(n, n_0, x_0)\| \leq \frac{M e^{(\alpha_1+\alpha_2)} \|x_0\|}{1^{-\gamma} \sqrt{1 - (M \|x_0\|)^{\gamma-1} e^{(\gamma-1)(\alpha_1+\alpha_2)} \alpha_3(\gamma-1)}}. \quad (23)$$

We observe that estimate (23) is valid only if (22) holds. Moreover, the radius of attraction (22) and estimate (23) depend directly from the series α_i of the coefficients $\lambda_i(n)$, ($i = 1, 2, 3$).

Example 2. Consider the delay difference equation

$$y(n+1) = y(n) + \sum_{i=1}^3 \lambda_i(n) y(n - \tau_i)^{n_i}, \quad (24)$$

where $n \geq \max\{\tau_1, \tau_2, \tau_3\}$, the constant delays τ_i ($i = 1, 2, 3$) are nonnegative integers and n_i ($i = 1, 2, 3$) are real numbers such that $1 < n_1 \leq n_2 \leq n_3$.

In view of the above assumptions, we have:

- (A) If we take $\omega_i(u) = u^{n_i}$, $i = 1, 2, 3$ then conditions (i) and (ii) of Theorem (1) are verified.
 (B) To see that condition (iii) of Theorem (1) is satisfied, we will carry out the following computations:

$$\varphi_0(c) = c, \quad \varphi_1(c) = [c^{1-n_1} + \alpha_1(1-n_1)]^{1/(1-n_1)}$$

for $0 < c < c_1$, where

$$c_1 = [\alpha_1(n_1 - 1)]^{1/(1-n_1)},$$

$$\varphi_2(c) = [c^{1-n_1} + \alpha_1(1-n_1)]^{(n_2-1)/(n_1-1)} + \alpha_2(1-n_2)]^{1/(1-n_1)}$$

for $0 < c < c_2$, where

$$c_2 = [\alpha_1(n_1 - 1) + (\alpha_2(n_2 - 1))^{(n_1-1)/n_2-1}]^{1/(1-n_1)}.$$

We have

$$\int_{\varphi_{i-1}(c)}^{\infty} \frac{ds}{\omega_i(s)} = \frac{1}{n_i - 1} (\varphi_{i-1}(c))^{1-n_i}, \quad i = 1, 2, 3, \quad \text{for } c \in \text{Domain of } \varphi_{i-1}.$$

Then, condition (iii) of Theorem (1) is equivalent to

$$\alpha_3 < \frac{1}{n_3 - 1} (\varphi_2(c))^{1-n_3}, \quad \text{or } \varphi_2(c) < \left(\frac{1}{\alpha_3(n_3 - 1)} \right)^{1/(n_3-1)}. \quad (25)$$

Since φ_i ($i = 1, 2, 3$) are monotone functions and by (13) $\lim_{c \rightarrow 0^+} \varphi_i(c) = 0$, hence, choosing c small enough, we get that (25) is satisfied. Solving the equation

$$\varphi_2(c^*) = \left(\frac{1}{\alpha_3(n_3 - 1)} \right)^{1/(n_3-1)}$$

we obtain

$$c^* = \{[\alpha_3(n_3 - 1)]^{(n_2-1)/(n_3-1)} + \alpha_2(n_2 - 1)]^{(n_1-1)(n_2-1)} + \alpha_1(n_1 - 1)\}^{1/(1-n_1)}.$$

(actually $c^* = \varphi_3^{-1}(\infty)$). Then, taking $c \leq c^*$, condition (iii) of Theorem 1 is satisfied.

Since the solutions of the difference equation $x(n+1) = x(n)$ are $x(n, n_0, x_0) = x_0$, then by Remark 1 the conditions of Theorem 1 are satisfied, and therefore we conclude that every solution $y(n, n_0, x_0)$ of Eq. (24) such that $\|x_0\| < c^*$, is defined and bounded on N_0 . Moreover, it follows from (14) that

$$\begin{aligned} \|y(n, n_0, x_0)\| &\leq \varphi_3(\|x_0\|) \\ &= [[[\|x_0\|^{1-n_1} + \alpha_1(1-n_1)]^{(n_2-1)/(n_1-1)} \\ &\quad + \alpha_2(1-n_2)]^{(n_3-1)/(n_2-1)} + \alpha_3(1-n_3)]^{1/(n_3-1)}, \end{aligned}$$

which implies the stability of the delay difference equation (24). As already pointed out in Example 1, the radius of attraction and the estimate of the solutions depend directly on the series α_i of the coefficients $\lambda_i(n)$, $i = 1, 2, 3$. \square

References

- [1] D. Bainov, P. Simeonov, Systems with impulsive effect, Stability, Theory and Applications, Ellis Horwood Limited, Series in Mathematics and Its Applications, 1989.
- [2] D. Bainov, P. Simeonov, Integral Inequalities and Applications, Kluwer Academic Publishers, Dordrecht, 1992.
- [3] G. Ladde, M. Sambandham, Variation of constants formula and error estimates to stochastic difference equations, J. Math. Phy. Sci. 22 (1988) 557–584.
- [4] V. Lakshmikantham, D. Trigiante, Theory of Difference Equations with Applications in Numerical Analysis, Academic Press, New York, 1988.
- [5] N. Luca, P. Talpalaru, Stability and asymptotic behavior of a class of discrete systems, Ann. Mat. Pure Appl. 112 (1977) 351–382.
- [6] R. Medina, Perturbations of nonlinear systems of difference equations, J. Math. Anal. Appl. 204 (1966) 545–553.
- [7] R. Medina, Stability and asymptotic behavior of difference equations, J. Comput. Appl. Math. 80 (1997) 17–30.
- [8] R. Medina, M. Pinto, Nonlinear discrete inequalities and stability of difference equations, World Sci. Ser. Appl. Anal. 3 (1994) 467–482.
- [9] M. Pinto, Asymptotic integration of the functional differential equation $y' = (t)a(t)y(t - r(t, y))$, J. Math. Anal. Appl. 175 (1993) 46–52.
- [10] Q. Sheng, R. Agarwal, On nonlinear variation of parameter methods for summary difference equations, Dyn. Systems Appl. 2 (1993) 227–242.